

THE BERGMAN REPRESENTATIVE MAP VIA A HOLOMORPHIC CONNECTION

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ABSTRACT. We show that the exponential map of the Bochner connection on the restricted holomorphic tangent bundle of a complex manifold admitting the positive-definite Bergman metric coincides with the inverse of Bergman's representative map. We also present a generalization of the Lu theorem, as an application.

1. INTRODUCTION

On a complex manifold equipped with the Bergman kernel and metric, the Bergman representative map, originally named as “the representative domain” by Stefan Bergman himself, is an important offshoot of the Bergman kernel form (cf. [14], Chapter 4). It is a special holomorphic map which is in a significant contrast with the exponential map of the Riemannian structure given by the real part of the Bergman metric; the Riemannian exponential map is almost never holomorphic. On the other hand, the representative map gives rise to a holomorphic Kähler normal coordinate system with respect to the Bergman metric. One of its best known features is that all holomorphic Bergman isometries become linear mappings in these representative coordinates. In spite of the difficulty that this map is not well-defined everywhere, this feature has been proven to be useful in many important works (see for instance, [23], [1], [29], [15], et al.). However, it was striking to us that no systematic study of this concept has yet been carried out.

The goal of this paper is therefore to provide a first step towards establishing a systematic treatise on the Bergman representative map. In particular, we present a construction of the torsion-free flat holomorphic affine connection on the holomorphic tangent bundle of an open dense subdomain of the given complex manifold, whose affine exponential map is the inverse to the representative map (Theorem 4.2). This yields a differential geometric interpretation of the Bergman representative map.

It is worth mentioning that our connection was discovered, at least partially, by several other authors in the articles preceding this paper, even though the information was scattered around in the papers such as [7] (much earlier than the others; in fact, Bochner constructed “normal” coordinates only, which can develop into the connection), [8], and [3]. It is also studied independently in [10] and [20] in relation to the holomorphic part of the Kähler metric connection (a symplectic geometric interpretation can be found in [22] and [27]). More notably, as the connection for the case

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of “bounded domains”, it was studied in [29] for a version of extension theorem for biholomorphic mappings. We hope that this paper shows these concepts in a unified viewpoint.

This paper is organized as follows: First, we briefly review fundamentals of Bergman geometry including the construction of the concept of the representative map. Then we present Bochner’s normal coordinate system for the real analytic Kähler manifolds and the affine connection. We would like to call it *the Bochner connection*. Then, we restrict ourselves to complex manifolds with the Bergman metric, and study the Bochner connection.

We demonstrate that our study is not without a notable main statement (Theorem 6.1); we present a generalization of the theorem by Lu Qi-Keng [23], which says that a bounded domain in \mathbb{C}^n whose Bergman metric is complete and of a constant holomorphic sectional curvature is biholomorphic to the unit ball. We were able to generalize this to the case of bounded domains with a *pole of the Bochner connection* such as a circular domain or a homogeneous domain.

2. FUNDAMENTALS OF BERGMAN GEOMETRY

2.1. The Bergman kernel and metric for a bounded domain in \mathbb{C}^n . Let Ω be a bounded domain in \mathbb{C}^n and $K(z, \bar{w})$ the Bergman kernel of Ω . Since $K(z, \bar{z}) > 0$, the Bergman metric

$$g_\Omega(z) = \sum_{j,k=1}^n g_{j\bar{k}}(z) dz_j \otimes d\bar{z}_k \quad \text{with} \quad g_{j\bar{k}}(z) = g_{j\bar{k}}(z, \bar{z}) := \frac{\partial^2 \log K(z, \bar{z})}{\partial z_j \partial \bar{z}_k}$$

is well-defined. In fact, the following result was proved by Bergman himself [2]:

Theorem 2.1 (Bergman). *The Bergman metric g_Ω is positive-definite at every $z \in \Omega$.*

Remark 2.2. Note that g_Ω is a Kähler metric. The transformation formula for the Bergman kernel function (under biholomorphisms) implies that every biholomorphism between bounded domains is an isometry with respect to the Bergman metric.

2.2. The Bergman representative map. Let p be a point of Ω . Since $K(p, \bar{p}) > 0$, there is a neighborhood of p such that $K(z, \bar{w}) \neq 0$ for all z, w in that neighborhood. Denote by $g^{\bar{k}j}(p)$ the (k, j) -th entry of the inverse matrix of $(g_{j\bar{k}}(p))$.

Definition 2.3. The *Bergman representative map* at p is defined by

$$\text{rep}_p(z) = (\zeta_1(z), \dots, \zeta_n(z)),$$

where:

$$\zeta_j(z) := g^{\bar{k}j}(p) \left\{ \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K(z, \bar{w}) - \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K(w, \bar{w}) \right\}.$$

Since $\frac{\partial \zeta_k}{\partial z_l} \Big|_{z=p} = \delta_{lk}$, this map defines a holomorphic local coordinate system at p . Another special feature is in the following theorem by Bergman himself.

Theorem 2.4 (Bergman). *If $f : \Omega \rightarrow \tilde{\Omega}$ is a biholomorphic mapping of bounded domains, then $\text{rep}_{f(p)} \circ f \circ \text{rep}_p^{-1}$ is \mathbb{C} -linear.*

The original proof of this by Bergman was via a direct computation using the transformation formula. On the other hand, a differential geometric proof using the Bochner connection will be presented in Section 4 (see Theorem 4.2 as well as Remark 4.3). Since the Bergman kernel and metric can be defined for complex manifolds [21], this geometric explanation applies to the case of complex manifolds.

2.3. The Bergman kernel form on a complex manifold. Let M be an n -dimensional complex manifold and $A^2(M)$ the space of holomorphic n -forms f on M satisfying

$$\left| \int_M f \wedge \bar{f} \right| < \infty.$$

Let $\{\phi_0, \phi_1, \phi_2, \dots\}$ be a complete orthonormal basis for the Hilbert space $A^2(M)$ and \bar{M} the complex manifold conjugate to M . Define the holomorphic $2n$ -form on $M \times \bar{M}$ by

$$K(z, \bar{w}) = \sum_{j=0}^{\infty} \phi_j(z) \wedge \overline{\phi_j(w)}.$$

This construction is independent of the choice of orthonormal basis. Using the diagonal embedding $\iota : M \hookrightarrow M \times \bar{M}$, defined by $\iota(z) = (z, \bar{z})$, and the natural identification of M with $\iota(M)$, $K(z, \bar{z})$ can be considered as a $2n$ -form on M . This is called the *Bergman kernel form* of M .

Consider the case that the Bergman kernel form is non-zero at any point of M . In a local coordinate system $(U, (z_1, \dots, z_n))$, the Bergman kernel form can be written as

$$K(z, \bar{z}) = K_U^*(z, \bar{z}) dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n,$$

where $K_U^*(z, \bar{z})$ is a well-defined function on U . Set

$$ds_M^2 := \sum_{j,k=1}^n g_{j\bar{k}}(z) dz_j \otimes d\bar{z}_k = \sum_{j,k=1}^n \frac{\partial^2 \log K_U^*(z, \bar{z})}{\partial z_j \partial \bar{z}_k} dz_j \otimes d\bar{z}_k.$$

This is independent of the choice of local coordinate system. When the matrix $G(z) := (g_{j\bar{k}}(z))$ is positive-definite for each $z \in M$, ds_M^2 is called the *Bergman metric* of M .

2.4. Bergman representative coordinates. From now on, suppose that M is a complex manifold which possesses the Bergman metric. (In fact, many complete non-compact Kähler manifold with negative curvature admit the Bergman metric [16], Theorem H). In a local coordinate system $(U \times \bar{V}, (z_1, \dots, z_n, \bar{w}_1, \dots, \bar{w}_n))$ for $M \times \bar{M}$,

$$K(z, \bar{w}) = K_{U \times \bar{V}}^*(z, \bar{w}) dz_1 \wedge \dots \wedge dz_n \wedge d\bar{w}_1 \wedge \dots \wedge d\bar{w}_n,$$

where $K_{U \times \bar{V}}^*(z, \bar{w})$ is a well-defined function on $U \times \bar{V}$. Given a point $\bar{p} \in \bar{V}$, define the following holomorphic coordinate system centered at p (cf. [9], [14], and [12]).

Definition 2.5. The *Bergman representative coordinate system* at p is defined by

$$\text{rep}_p(z) = (\zeta_1(z), \dots, \zeta_n(z)),$$

where:

$$\zeta_j(z) := g^{\bar{k}j}(p) \left\{ \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K_{U \times \bar{V}}^*(z, \bar{w}) - \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K_{V \times \bar{V}}^*(w, \bar{w}) \right\}.$$

Remark 2.6. The above construction is independent of the choice of a coordinate system $(U, (z_1, \dots, z_n))$ for M . But it depends on the choice of local coordinate system $(V, (w_1, \dots, w_n))$. Note that rep_p extends to a global function, well-defined on the whole of M except for the analytic variety $Z_0^p := \{z \in M : K(z, \bar{p}) = 0\}$.

3. BOCHNER'S NORMAL COORDINATES AND CONNECTION

For the real analytic Kähler manifolds, Bochner constructed the Kähler normal coordinate system, a version of the representative coordinate system from the Kähler potential [7]. This normal coordinate system is strongly related to the exponential map of the Kähler metric [10]. We feel that this relation can be better explained via the language of vector bundles and connections [20]. Therefore, we reorganize this information, scattered in the literature.

3.1. Bochner's normal coordinates. Suppose that M is a Kähler manifold with the real analytic Kähler metric g . In [7], a Kähler normal coordinate system is defined as follows:

Proposition 3.1 (Bochner's normal coordinates). *Given $p \in M$, there exist holomorphic coordinates $(\zeta_1, \dots, \zeta_n)$, unique up to unitary linear transformations satisfying*

- (i) $\zeta(p) = 0$,
- (ii) $g_{j\bar{k}}(p) = \delta_{jk}$,
- (iii) $dg_{j\bar{k}}(p) = 0$,
- (iv) $\frac{\partial^I g_{j\bar{k}}}{\partial \zeta_1^{i_1} \dots \partial \zeta_n^{i_n}}(p) = 0$, for all $I \geq 0$ and $i_1 + \dots + i_n = I$.

In [3], Bochner's coordinate system was rediscovered in the context of mathematical physics. There, the Bochner coordinates were called the *canonical coordinates*. Their result is

Proposition 3.2 (Bershadsky, Cecotti, Ooguri and Vafa [3]). *Bochner's normal coordinates $(\zeta_1, \dots, \zeta_n)$ can be expressed in terms of the Kähler potential $\psi(z, \bar{z})$:*

$$\zeta_j(z) = \sqrt{g}^{\bar{k}j}(p) \left\{ \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \psi(z, \bar{w}) - \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \psi(w, \bar{w}) \right\},$$

where $\sqrt{g}^{\bar{k}j}(p)$ is as follows: Since $G(p) := (g_{j\bar{k}}(p))$ is a positive-definite Hermitian matrix, there exists a matrix A such that $G(p) = A\bar{A}^t$. Denote by $\sqrt{g}^{\bar{k}j}(p)$ the (k, j) -th entry of the inverse matrix of A .

Corollary 3.3. *Bochner's normal coordinate system for a manifold with the Bergman metric is the same as the Bergman representative coordinate system of the Kähler potential $\log K(z, \bar{z})$ up to the normalization factor $\sqrt{g}^{\bar{k}j}(p)$.*

We explain how to separate the holomorphic part from the Riemannian exponential map and show that it coincides with the inverse map of the Bochner normal coordinate system. Our exposition follows those of [10], [11], and [20].

3.2. The holomorphic exponential map. Let M be a real-analytic Kähler manifold. The construction of the holomorphic exponential map from the Riemannian exponential map $\exp_p : T_p M \rightarrow M$ consists of two steps: **(1) complexification**, **(2) restriction**.

Step 1. Complexification (Polarization). Note that the real-analytic manifold M of real dimension n can be embedded to become a totally real submanifold of the complex manifold $\mathbb{C}M$ of complex dimension n .

Theorem 3.4 (Whitney-Bruhat [30]). *Every real analytic manifold M can be embedded to become a totally real submanifold of a complex manifold. This embedding is unique in the sense that, if $\iota_1 : M \hookrightarrow \mathbb{C}M_1$ and $\iota_2 : M \hookrightarrow \mathbb{C}M_2$ are such embeddings, then there exist neighborhoods U_1 and U_2 of M in $\mathbb{C}M_1$ and $\mathbb{C}M_2$ respectively, and a biholomorphism $f : U_1 \rightarrow U_2$ such that $\iota_2 = f \circ \iota_1$.*

Take the complexification $T_p M \hookrightarrow T_p^{\mathbb{C}} M$ and the diagonal embedding $\iota : M \hookrightarrow M \times \overline{M}$. Then, apply the following lemma to the exponential map $\exp_p : T_p M \rightarrow M$.

Lemma 3.5. *Let M and N be totally real submanifolds of complex manifolds $\mathbb{C}M$ and $\mathbb{C}N$, and $f : M \rightarrow N$ a real-analytic diffeomorphism. Then there are neighborhoods U and V of M and N , and a unique holomorphic map $f^{\mathbb{C}} : U \rightarrow V$ extending f .*

Denote by $\exp_p^{\mathbb{C}}$ the unique holomorphic extension of \exp_p .

Step 2. Restriction. Use the decomposition $T_p^{\mathbb{C}} M \cong T_p' M \oplus T_p'' M$ where $T_p' M$: the holomorphic tangent space and $T_p'' M$: the anti-holomorphic tangent space. Then restrict the complexified map $\exp_p^{\mathbb{C}}$ to $T_p' M$.

Definition 3.6. The restriction map $\exp_p^{\mathbb{C}}|_{T_p' M}(\zeta) := \exp_p^{\mathbb{C}}(\zeta, 0)$ is called the *holomorphic exponential map*.

We remark that the above definition is the same as the following definition, appeared first in [10].

Definition 3.7. Take the power series expansion of the exponential map of the Kähler metric $\exp_p : T_p' M \oplus T_p'' M \rightarrow M$ and the decomposition

$$\exp_p(\zeta, \bar{\zeta}) = f(\zeta) + g(\zeta, \bar{\zeta}),$$

on some neighborhood of 0, where f is holomorphic in ζ and g is the sum of all monomials which are not holomorphic in ζ . Then *the holomorphic part* of the exponential map at p is defined to be

$$\exp_h(\zeta) := f(\zeta).$$

3.3. The Bochner connection. We present the construct of the holomorphic affine connection, whose affine exponential map is the *holomorphic exponential map* \exp_p . We also show that \exp_p is the same as the inverse to the Bochner normal coordinate system, using the affine geodesic equations of the connection.

Theorem 3.8 (Kapranov [20]). *There exists a holomorphic affine connection $\nabla^{\mathbb{C}}$ on $T'(M \times \overline{M})$, defined over a neighborhood of $\iota(M)$, whose affine exponential map is $\exp_p^{\mathbb{C}}$. The restriction of $\nabla^{\mathbb{C}}$ to $T'_p M$ is also a holomorphic affine connection, defined over a neighborhood of p in M . The affine exponential map of $\nabla^{\mathbb{C}}|_{T'_p M}$ is \exp_p .*

Proof. Let ∇ be the Kähler connection, defined by the Christoffel symbols $\Gamma_{kl}^j(z, \bar{z}) = \frac{\partial g_{k\bar{m}}(z, \bar{z})}{\partial z_l} g^{\bar{m}j}(z, \bar{z})$. Denote by $\nabla^{\mathbb{C}}$ the analytic continuation (complexification) of ∇ . Then $\nabla^{\mathbb{C}}$ is an affine connection, defined by the coefficients of the connection 1-form:

$$\Gamma_{kl}^j(z, \bar{w}) = \frac{\partial g_{k\bar{m}}(z, \bar{w})}{\partial z_l} g^{\bar{m}j}(z, \bar{w}),$$

where (z, \bar{w}) are holomorphic coordinates for $M \times \overline{M}$. Moreover, its affine exponential map is the same as $\exp_p^{\mathbb{C}}$, since the complexification of \exp_p is unique.

To prove the second statement, take the decomposition

$$T'_{(p, \bar{p})}(M \times \overline{M}) = T'_p M \oplus T'_{\bar{p}} \overline{M},$$

and restrict $\nabla^{\mathbb{C}}$ to $T'_p M$. Then this is a holomorphic affine connection on $T'_p M$, defined only in some neighborhood of p in M . The affine exponential map of this connection is the holomorphic exponential map \exp_p . \square

From now on, we denote $\nabla^{\mathbb{C}}|_{T'_p M}$ by ∇^p , and call it the *Bochner connection* at p . The following lemma shows the affine geodesic equations for the holomorphic exponential map \exp_p .

Lemma 3.9. *The geodesics of the Bochner connection ∇^p emanating from p in the initial direction $\zeta \in T'_p M$ satisfies the following system of second order ODE:*

$$(3.1) \quad \begin{cases} \frac{d^2 z_j(t)}{dt^2} + \Gamma_{kl}^j(z(t), \bar{p}) \frac{dz_k(t)}{dt} \frac{dz_l(t)}{dt} = 0, \\ z(0) = p, \quad \frac{dz_j}{dt}(0) = \zeta_j. \end{cases}$$

Proof. The curve $\exp_p^{\mathbb{C}}(\zeta t, \bar{\xi} t)$, constructed by the affine exponential map of $\nabla^{\mathbb{C}}$ satisfies

$$(3.2) \quad \begin{cases} \frac{d^2 z_j(t)}{dt^2} + \Gamma_{kl}^j(z(t), \bar{w}(t)) \frac{dz_k(t)}{dt} \frac{dz_l(t)}{dt} = 0, \\ z(0) = p, \quad \frac{dz_j}{dt}(0) = \zeta_j, \end{cases}$$

and

$$(3.3) \quad \begin{cases} \frac{d^2 \bar{w}_j(t)}{dt^2} + \Gamma_{kl}^{\bar{j}}(z(t), \bar{w}(t)) \frac{d\bar{w}_k(t)}{dt} \frac{d\bar{w}_l(t)}{dt} = 0, \\ \bar{w}(0) = \bar{p}, \quad \frac{d\bar{w}_j}{dt}(0) = \bar{\xi}_j, \end{cases}$$

where $(\zeta, \bar{\xi}) \in T'_p M \oplus T''_p M = \mathbb{C}T_p M$. It suffices to let $\xi \equiv 0$, since the solution of (3.3) becomes the constant map $(w_1, \dots, w_n) \equiv (p_1, \dots, p_n)$. \square

Using the above lemma, we prove the following proposition, appeared first in [20].

Proposition 3.10. *The inverse to the holomorphic exponential map at p of the real analytic Kähler metric is the Bochner normal coordinate system at p , up to unitary linear transformations.*

Proof. Let φ be the inverse to the Bochner normal coordinate system and $\tilde{\gamma}(t)$ the curve in M given by $\tilde{\gamma}(t) = \varphi(vt)$ where $v \in \mathbb{C}^n \cong T'_p M$. It is enough to show that $\tilde{\gamma}(t) = (z_1(t), \dots, z_n(t))$ satisfies (3.1). By the definition of the normal coordinates, we obtain $\tilde{\gamma}(0) = \varphi(0) = p$ and

$$(3.4) \quad \frac{\partial \zeta_k}{\partial z_l} = g^{\bar{j}k}(p) g_{l\bar{j}}(z, \bar{p}), \quad \frac{\partial z_k}{\partial \zeta_r} = g_{r\bar{\lambda}}(p) g^{\bar{\lambda}k}(z, \bar{p}).$$

Since $\frac{\partial \zeta_k}{\partial z_l} \big|_{z=p} = \delta_{lk}$, $\tilde{\gamma}'(0) = (\frac{dz_1}{dt}(0), \dots, \frac{dz_n}{dt}(0)) = (\frac{d\zeta_1}{dt}(0), \dots, \frac{d\zeta_n}{dt}(0)) = v$. Then the holomorphicity of the Bochner normal coordinates implies that

$$\begin{aligned} \frac{d^2 z_j(t)}{dt^2} + \Gamma_{kl}^j(\tilde{\gamma}(t), \bar{p}) \frac{dz_k(t)}{dt} \frac{dz_l(t)}{dt} \\ = \frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} \frac{d\zeta_r}{dt} \frac{d\zeta_s}{dt} + \Gamma_{kl}^j(\tilde{\gamma}(t), \bar{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{d\zeta_r}{dt} \frac{\partial z_l}{\partial \zeta_s} \frac{d\zeta_s}{dt} \\ = \left\{ \frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} + \Gamma_{kl}^j(\tilde{\gamma}(t), \bar{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{\partial z_l}{\partial \zeta_s} \right\} v_r v_s. \end{aligned}$$

Thus it suffices to show that the following analytic differential equations hold:

$$\frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} + \Gamma_{kl}^j(\tilde{\gamma}(t), \bar{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{\partial z_l}{\partial \zeta_s} = 0.$$

The matrix equation $d(A \cdot A^{-1}) = 0$ and the equation (3.4) yield

$$\begin{aligned} \frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} &= g_{r\bar{\lambda}}(p) \frac{\partial g^{\bar{\lambda}j}(z, \bar{p})}{\partial z_l} \frac{\partial z_l}{\partial \zeta_s} \\ &= -g_{r\bar{\lambda}}(p) g^{\bar{\lambda}k}(z, \bar{p}) \frac{\partial g_{k\bar{m}}(z, \bar{p})}{\partial z_l} g^{\bar{m}j}(z, \bar{p}) \frac{\partial z_l}{\partial \zeta_s} \\ &= -\frac{\partial z_k}{\partial \zeta_r} \frac{\partial g_{k\bar{m}}(z, \bar{p})}{\partial z_l} g^{\bar{m}j}(z, \bar{p}) \frac{\partial z_l}{\partial \zeta_s}. \end{aligned}$$

Therefore, we arrive at

$$\frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} + \Gamma_{kl}^j(\tilde{\gamma}(t), \bar{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{\partial z_l}{\partial \zeta_s} = 0.$$

\square

Remark 3.11. For a bounded domain with the Bergman metric, it is known that the above analytic equations hold for the Bergman representative map [24]. This is, of course, strongly analogous to the analysis on the flow of vector fields in the context of Riemannian geometry.

4. THE BOCHNER CONNECTION ON A MANIFOLD WITH THE BERGMAN METRIC

Let M be a complex manifold with the Bergman metric and p a point in M . Since the Bergman metric is a real-analytic Kähler metric, the Bochner connection can be constructed in an open neighborhood of p as in Section 3.3. On the other hand, we show that the Bochner connection actually extends to the whole manifold except possibly for an analytic variety.

4.1. The extended Bochner connection. Suppose that M is a complex manifold which possesses the Bergman metric. Recall that in a local coordinate system $(U \times \bar{V}, (z_1, \dots, z_n, \bar{w}_1, \dots, \bar{w}_n))$ for $M \times \bar{M}$, the Bergman kernel form is

$$K(z, \bar{w}) = K_{U \times \bar{V}}^*(z, \bar{w}) dz_1 \wedge \dots \wedge dz_n \wedge d\bar{w}_1 \wedge \dots \wedge d\bar{w}_n,$$

where $K_{U \times \bar{V}}^*(z, \bar{w})$ is a well-defined function on $U \times \bar{V}$. Define the tensor on $M \times \bar{M}$ by

$$G(z, \bar{w}) := \sum_{j,k=1}^n g_{j\bar{k}}(z, \bar{w}) dz_j \otimes d\bar{w}_k = \sum_{j,k=1}^n \frac{\partial^2 \log K_{U \times \bar{V}}^*(z, \bar{w})}{\partial z_j \partial \bar{w}_k} dz_j \otimes d\bar{w}_k.$$

Let $(\tilde{U} \times \tilde{\bar{V}}, (\tilde{z}_1, \dots, \tilde{z}_n, \tilde{\bar{w}}_1, \dots, \tilde{\bar{w}}_n))$ be another coordinate system. Then, in $(U \times \bar{V}) \cap (\tilde{U} \times \tilde{\bar{V}})$, the following transformation formula

$$(4.1) \quad K_{U \times \bar{V}}^*(z, \bar{w}) = K_{\tilde{U} \times \tilde{\bar{V}}}^*(\tilde{z}, \tilde{\bar{w}}) \det J_{\tilde{U}}^{\tilde{U}}(z) \overline{\det J_{\tilde{V}}^{\tilde{V}}(w)}$$

holds, where $J_{\tilde{U}}^{\tilde{U}}(z) = \left(\frac{\partial \tilde{z}_k}{\partial z_j} \right)_{n \times n}$ and $J_{\tilde{V}}^{\tilde{V}}(w) = \left(\frac{\partial \tilde{w}_k}{\partial w_j} \right)_{n \times n}$. In terms of matrices,

$$(4.2) \quad G_{U \times \bar{V}}(z, \bar{w}) = J_{\tilde{U}}^{\tilde{U}}(z) \cdot \tilde{G}_{\tilde{U} \times \tilde{\bar{V}}}(\tilde{z}, \tilde{\bar{w}}) \cdot \overline{J_{\tilde{V}}^{\tilde{V}}(w)}^t$$

where $G_{U \times \bar{V}}(z, \bar{w}) = \left(\frac{\partial^2 \log K_{U \times \bar{V}}^*(z, \bar{w})}{\partial z_j \partial \bar{w}_k} \right)_{n \times n}$ and $\tilde{G}_{\tilde{U} \times \tilde{\bar{V}}}(\tilde{z}, \tilde{\bar{w}}) = \left(\frac{\partial^2 \log K_{\tilde{U} \times \tilde{\bar{V}}}^*(\tilde{z}, \tilde{\bar{w}})}{\partial \tilde{z}_j \partial \tilde{\bar{w}}_k} \right)_{n \times n}$.

Given a point $\bar{p} \in \bar{M}$, define the analytic varieties

$$Z_0^p := \{z \in M : K(z, \bar{p}) = 0\} \quad \text{and} \quad Z_1^p := \{z \in M - Z_0^p : \det(G(z, \bar{p})) = 0\}.$$

Lemma 4.1. *If $f : M \rightarrow \tilde{M}$ is a biholomorphism with $q = f(p)$, then it satisfies*

- (1) $f(Z_0^p) = \tilde{Z}_0^q$,
- (2) $f(Z_1^p) = \tilde{Z}_1^q$,
- (3) $f(M^p) = \tilde{M}^q$,

where $M^p := M - (Z_0^p \cup Z_1^p)$ and $\tilde{M}^q := \tilde{M} - (\tilde{Z}_0^q \cup \tilde{Z}_1^q)$.

Proof. The transformation formulae (4.1) and (4.2) prove that these sets are well-defined and invariant under biholomorphisms. \square

Let $T'M^p$ be the holomorphic tangent bundle over M^p . Then,

Theorem 4.2. *There exists a holomorphic affine connection ∇^p on $T'M^p$ satisfying:*

- (1) ∇^p is locally flat, i.e. the torsion and curvature of ∇^p are zero.

- (2) (3.1) are the affine geodesic equations for ∇^p .
- (3) $f_*(\nabla_X^p Y) = \nabla_{\tilde{X}}^q \tilde{Y}$ for all $X, Y \in T'M^p$ where $\tilde{X} = f_*(X)$, $\tilde{Y} = f_*(Y)$ and f is the same as in the preceding lemma.

Proof. The proof is essentially the same as that of the case of bounded domains (cf. [29]).

Define the connection 1-forms as follows: Note that $G := G_{U \times \bar{V}}(z, \bar{p})$ is an invertible holomorphic $(n \times n)$ -matrix on $U \cap M^p$ so that G^{-1} is well-defined on $U \cap M^p$. Define the $(n \times n)$ -matrix ω of holomorphic 1-forms by $\omega := \partial G \cdot G^{-1}$, locally defined on $U \cap M^p$. In other words,

$$\omega_i^j(z) = \Gamma_{ik}^j(z, \bar{p}) dz_k = \frac{\partial g_{i\bar{m}}(z, \bar{p})}{\partial z_k} g^{\bar{m}j}(z, \bar{p}) dz_k.$$

Since $\partial G = \omega \cdot G$, the transformation formula (4.2) yields the *transformation rule for connection 1-forms*:

$$(4.3) \quad \omega \cdot J = \partial J + J \cdot \tilde{\omega}.$$

To show (1), observe that ∇^p is torsion-free, because $\frac{\partial}{\partial z_k} g_{i\bar{j}} = \frac{\partial}{\partial z_i} g_{k\bar{j}}$. Moreover, its curvature form $\Omega := d\omega - \omega \wedge \omega$ is also zero, because $G := (g_{j\bar{k}}(z, \bar{p}))$ is holomorphic. More precisely,

$$\begin{aligned} d(\partial G \cdot G^{-1}) - (\partial G \cdot G^{-1}) \wedge (\partial G \cdot G^{-1}) &= \partial(\partial G \cdot G^{-1}) + \partial G \wedge \partial G^{-1} \cdot G \cdot G^{-1} \\ &= -\partial G \wedge \partial G^{-1} + \partial G \wedge \partial G^{-1} = 0. \end{aligned}$$

Now, (2) follows immediately from the construction, and (3) follows by (4.3). \square

Remark 4.3. The last statement in Theorem 4.2 implies the \mathbb{C} -linearity of the Bergman representative coordinates as follows: Since geodesics are straight lines in this coordinate system, $\text{rep}_{f(p)} \circ f \circ \text{rep}_p^{-1}$ maps straight lines to straight lines. Thus it is \mathbb{R} -linear. Since the representative map is holomorphic, this is \mathbb{C} -linear. This is the geometric proof of Theorem 2.4, promised in Section 2.2.

It is possible to find the formula of the inverse to the affine exponential map of ∇^p not only at p but also at an arbitrary point $q \in M^p$. The proof is the same as that of Theorem 3.10.

Proposition 4.4. *Denote by exp_q the affine exponential map of ∇^p at q . Let $(\zeta_1, \dots, \zeta_n)$ be the coordinate system for the holomorphic tangent space at q , $T'_q M^p$. Then, in the local coordinate neighborhood $(U, (z_1, \dots, z_n))$ containing q ,*

$$\text{exp}_q^{-1}(z) = (\zeta_1(z), \dots, \zeta_n(z)),$$

where:

$$\zeta_j(z) = \sqrt{g^{\bar{k}j}}(q, \bar{p}) \left\{ \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K_{U \times \bar{V}}^*(z, \bar{w}) - \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K_{U \times \bar{V}}^*(q, \bar{w}) \right\}.$$

Remark 4.5. The above proposition implies that exp_q^{-1} is a linear transform of exp_p^{-1} . Therefore M^p can be covered by copies (by linear maps) of the representative coordinates. This can be explained by the concept of affine structures in [25]. This concept will be introduced in the following subsection.

Remark 4.6. For a real-analytic Kähler manifold, it is known that such an affine structure exists on a local neighborhood of a given point [24].

4.2. Affine structure of M^p . The proof of Theorem 3.10 also implies the following

Proposition 4.7. *Let U be a local neighborhood of p and V a local neighborhood of 0 such that $\exp_p^{-1} : U \rightarrow V$ is biholomorphic. Take any straight line l in V (not necessarily passing through p). Then $\exp_p(l)$ is a geodesic of ∇^p .*

This proposition follows immediately, understanding the affine structures as follows:

Definition 4.8. Let X be a complex manifold of dimension n and $\mathcal{M} = \{U_i, \phi_i\}_{i \in I}$ the maximal atlas. A subset $\mathcal{A} = \{U_j, \phi_j\}_{j \in J}$, $J \subset I$, of \mathcal{M} is called an *affine atlas* of X if all transition maps are complex affine transformation of \mathbb{C}^n . We say that each maximal affine atlas defines a complex *affine structure* of X .

Theorem 4.9 (Gunning [17], Matsushima [25]). *There is a one-to-one correspondence between the set of all complex affine structures on a complex manifold X and the set of all locally flat holomorphic affine connections on X .*

Remark 4.10. For any $x, y \in M^p$, $\exp_y \circ \exp_x^{-1}$ is an affine transformation of \mathbb{C}^n . Thus M^p has a complex affine structure and the Bochner connection ∇^p is the corresponding locally flat holomorphic affine connection.

4.3. Geodesics of ∇^p . The behavior of geodesics of ∇^p played an important role in the proof of the following theorem, which generalizes Fefferman's extension theorem.

Theorem 4.11 (Webster [29]). *Let $f : \Omega \rightarrow \tilde{\Omega}$ be a biholomorphism between bounded domains with smooth boundaries. Suppose that their Bergman kernels are smooth up to the boundaries. Then f extends smoothly to a dense open subset of $\partial\Omega$.*

Sketch of the proof. Since the kernel is smooth up to the boundary, ∇^p extends to a dense open subset of the boundary in the sense that, it is defined over the boundary except for limit points of the variety $Z_0^p \cup Z_1^p$. Thus the Bochner normal coordinate system, i.e. the Bergman representative coordinate system, extends over the boundary so that the corresponding geodesics extend through the boundary points. Then the result follows. \square

Notice that the incompleteness of ∇^p has played an important special role in this proof. On the other hand, one might expect to find a suitable Kähler metric, compatible to ∇^p . But this is impossible because the exponential map of a Kähler metric is holomorphic if and only if the metric is flat (the Euclidean metric). However, using the image of geodesics under $\text{rep}_p = \exp_p^{-1}$, it is possible to define a distance between two points in a connected manifold M^p .

Definition 4.12 (Intrinsic distance). Let M be a connected manifold with the Bergman metric and $\mathcal{A} = \{U_i, \phi_i\}_{i \in I}$ the affine structure of M^p , given by the Bochner normal coordinate system. If $x, y \in U_i$ for some U_i , then define $\delta^p(x, y)$ to be the euclidean norm of the vector

$$\left(\dots, \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K(x, \bar{w}) - \frac{\partial}{\partial \bar{w}_k} \Big|_{w=p} \log K(y, \bar{w}), \dots \right).$$

In general, if x, y are arbitrary points in M^p , then define the *intrinsic distance* by

$$d^p(x, y) := \inf \sum_{j=1}^N \delta^p(p_{j-1}, p_j),$$

where the infimum is taken over all possible partitions with $x = p_0, \dots, p_N = y$.

This is well-defined since there always exists a broken geodesic between two points in the connected affine manifold (M^p, ∇^p) .

Remark 4.13. For the symmetry $d^p(x, y) = d^p(y, x)$, we do not use the normalization factor of the Bochner normal coordinate system. Although d^p is not a biholomorphic invariant, its finiteness between two points is a biholomorphic invariant.

The following theorem shows the relation between the intrinsic distance d^p and the analytic variety Z_0^p .

Theorem 4.14. *If $q \in Z_0^p$, then there are no geodesics connecting to q with the intrinsic distance finite.*

Proof. Suppose that there exists a geodesic connecting q with the intrinsic distance finite. Then the Bochner normal coordinate system is well-defined at q . Note that $\frac{\partial}{\partial \bar{w}_k} K(q, \bar{w}) / K(q, \bar{w})$ is an anti-holomorphic function in w for each k . Fix an index k and then in the w_k -section, this is an one-variable function. Since $K(q, \bar{p}) = 0$, it has a simple pole at p so that the value must diverge. This implies that δ^p diverges, which contradicts the finiteness of the intrinsic distance. \square

5. ON THE VARIETY $Z_0^p \cup Z_1^p$ AND THE SKWARCZYNSKI ANNULUS

5.1. On the variety Z_0^p . Let Ω be a bounded domain in \mathbb{C}^n and $K(z, \bar{w})$ the Bergman kernel of Ω . Fix a point $p \in \Omega$. Then the Bochner connection ∇^p can be constructed over $\Omega^p = \Omega - (Z_0^p \cup Z_1^p)$. Whether Z_0^p is empty is related to the well-known Lu Qi-Keng conjecture; a bounded domain Ω is called a *Lu Qi-Keng domain* if Z_0^p is empty for each point $p \in \Omega$.

Remark 5.1. It was anticipated in 1960's that every bounded domain should be a Lu Qi-Keng domain, called the Lu Qi-Keng conjecture. However, many counterexamples have been discovered ([28], [5, 6], and others.), and in contrast, many domains are Lu Qi-Keng.

Example (Skwarczynski's annulus) Let $A := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$. Then the Bergman kernel of A is

$$K(z, \bar{w}) = \frac{\wp(\log z \bar{w}) + \frac{\eta_1}{\omega_1}}{\pi z \bar{w}},$$

where \wp is the Weierstrass elliptic function with half periods $\omega_1 = \log(1/r)$, $\omega_2 = \pi i$ and η_1 is the increment of the Weierstrass zeta function ζ with respect to ω_1 .

Zeros of $K(z, \bar{p})$: Define $h(\lambda) := \wp(\log \lambda) + \frac{\eta_1}{\omega_1}$ on the set $\tilde{A} := \{\lambda \in \mathbb{C} : r^2 < |\lambda| < 1\}$. Then $K(z, \bar{w}) = \frac{h(\lambda)}{\pi \lambda}$, where $\lambda = z \bar{w}$. In [28], Skwarczynski proved that

- $h(\lambda)$ is real, $\forall \lambda \in \mathbb{R}$.
- For $r < e^{-2}$, there exists a point $\lambda \in \tilde{A}$ such that $h(\lambda) = 0$.

Later, the above result could be improved as follows (cf. [4], Theorem 3.4):

- $h(-1) = h(-r^2) < 0$ and $h(-r) > 0$.
- For $r < 1$, there exist only two solutions λ_1, λ_2 of the equation $h(\lambda) = 0$ in $\tilde{A} = \{\lambda \in \mathbb{C} : r^2 < |\lambda| < 1\}$, where $\lambda_2 \in (-1, -r)$ and $\lambda_1 \in (-r, -r^2)$.

Fix a point $p \in A$. The symmetry of the annulus allows to assume that $p \in (r, 1)$ on the real line. Let λ_1^p and λ_2^p be the solutions of the equation $h(z\bar{p}) = 0$ satisfying $\lambda_1 = \lambda_1^p \bar{p}$ and $\lambda_2 = \lambda_2^p \bar{p}$. Since $\lambda_2^p \in (-1/p, -r/p)$, $\lambda_1^p \in (-r/p, -r^2/p)$, the number of elements of the zero set $\{z \in A : K(z, \bar{p}) = 0\}$ depends on the location of p . For example, if p is close enough to 1 (or r), then there exists only one solution of the equation $K(z, \bar{p}) = 0$ in the annulus A , located in $(-1, -r)$ (or $(-r, -r^2)$).

Geodesics of the Bochner connection ∇^p : Recall that the exponential map of the Bochner connection ∇^p is the inverse map of rep_p . A simple computation shows that

$$\text{rep}_p(z) = C_1 \cdot \frac{\wp'(\log z\bar{p})}{\wp(\log z\bar{p}) + \frac{\eta_1}{\omega_1}} + C_2,$$

where C_1 and C_2 are constants, and \wp' is the first derivative of the Weierstrass elliptic function \wp . Then rep_p is also an elliptic function that shares the same periods with \wp and possesses three simple poles. Since the image of any elliptic function contains \mathbb{C} , the pre-image of each straight line consists of three curves (counting multiplicity) in the annulus $\{z \in \mathbb{C} : r^2 < |z\bar{p}| < 1\}$.

Note that the geodesics of ∇^p are the images of straight lines by the holomorphic exponential map $\text{exp}_p = \text{rep}_p^{-1}$. Therefore, it is enough to study the pre-images of straight lines by the elliptic function $f(\lambda) := \frac{\wp'(\lambda)}{\wp(\lambda) + c}$ with the lattice $\Lambda = \{\mathbb{Z}(2\omega_1) + \mathbb{Z}(2\omega_2)\}$. Since $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, the function \wp and its derivative \wp' have rectangular lattice so that $\wp(z) = \overline{\wp(\bar{z})}$, $\wp'(z) = \overline{\wp'(\bar{z})}$. This implies that for all $t \in \mathbb{R}$,

- $\wp(2t\omega_j) = \overline{\wp(2t\omega_j)} = \overline{\wp(2t\omega_j)}$ and $\wp(\omega_i + t2\omega_j) = \overline{\wp(\omega_i + t2\omega_j)}$,
- $\wp'(2t\omega_j) = \pm \overline{\wp'(2t\omega_j)}$ and $\wp'(\omega_i + t2\omega_j) = \pm \overline{\wp'(\omega_i + t2\omega_j)}$.

Since $c = \frac{\eta_1}{\omega_1} \in \mathbb{R}$, we see that

- $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\mathbb{R}) \subset i\mathbb{R}$.
- $f(\mathbb{R} + \omega_2) \subset \mathbb{R}$ and $f(\omega_1 + i\mathbb{R}) \subset i\mathbb{R}$.

Therefore, $f^{-1}(\mathbb{C} - (\mathbb{R} \cup i\mathbb{R}))$ consists of 4 open sub-rectangles which are divided by ω_1 and ω_2 in the fundamental region $\{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 2\omega_1, 0 \leq \text{Im}(z) \leq \text{Im}(2\omega_2)\}$. This provides an approximate, but useful, information on the location of geodesics.

5.2. The relation between Z_0^p and Z_1^p . In the light of the Cheng-Yau conjecture [32] (solutions are announced in [13] and [18]), we present the following

Proposition 5.2. *Let Ω be a strictly pseudoconvex domain with smooth boundary. If the Bergman metric of Ω is Kähler-Einstein, then $Z_0^p = Z_1^p$ for every $p \in \Omega$.*

Proof. It turns out that $\frac{K(z, \bar{z})}{\det[g_{j\bar{k}}]}$ is a positive constant if the Bergman metric is Kähler-Einstein (cf. [13], Proposition 1.1). Therefore, the polarization gives the result. \square

On the other hand, note that

$$\det[G(z, \bar{p})] = \frac{\det[K(z, \bar{p}) \frac{\partial^2}{\partial z_i \partial \bar{w}_j} \Big|_{w=p} K(z, \bar{w}) - \frac{\partial}{\partial z_i} K(z, \bar{p}) \frac{\partial}{\partial \bar{w}_j} \Big|_{w=p} K(z, \bar{w})]}{K(z, \bar{p})^{2n}}.$$

Set

$$\widehat{Z}_1^p := \left\{ z \in \Omega : \det \left[K(z, \bar{p}) \frac{\partial^2}{\partial z_i \partial \bar{w}_j} \Big|_{w=p} K(z, \bar{w}) - \frac{\partial}{\partial z_i} K(z, \bar{p}) \frac{\partial}{\partial \bar{w}_j} \Big|_{w=p} K(z, \bar{w}) \right] = 0 \right\}.$$

Then we present:

Remark 5.3. If $n > 1$, then $Z_0^p \subset \widehat{Z}_1^p$, and the statement is false if $n = 1$ (cf. [12], Lemma 2.1). In particular, $Z_0^p \not\subset \widehat{Z}_1^p$ for the annulus in the complex plane. Therefore, it is natural to ask whether a domain of higher dimension satisfying $Z_0^p \subsetneq \widehat{Z}_1^p$ exists. Such a domain actually exists; when r is small enough, the product domain $A_r \times D$ has a point p such that $Z_0^p \subsetneq \widehat{Z}_1^p$, where $A_r := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is the annulus and D is the unit disk in \mathbb{C} . The proof is as in the following:

Let $K((z_1, z_2), (\bar{w}_1, \bar{w}_2)) = K_{A_r}(z_1, \bar{w}_1) K_D(z_2, \bar{w}_2)$ be the Bergman kernel of $A_r \times D$. Set $F_\Omega(z, \bar{w}) := \det[K_\Omega(z, \bar{w}) \frac{\partial^2}{\partial z_i \partial \bar{w}_j} K_\Omega(z, \bar{w}) - \frac{\partial}{\partial z_i} K_\Omega(z, \bar{w}) \frac{\partial}{\partial \bar{w}_j} K_\Omega(z, \bar{w})]$. Then

$$F_{A_r \times D}((z_1, z_2), (\bar{w}_1, \bar{w}_2)) = K_{A_r}(z_1, \bar{w}_1)^2 K_D(z_2, \bar{w}_2)^2 F_{A_r}(z_1, \bar{w}_1) F_D(z_2, \bar{w}_2).$$

Since $K_D(z_2, \bar{w}_2)^2 F_D(z_2, \bar{w}_2) \neq 0$ for all $(z_2, \bar{w}_2) \in D \times \bar{D}$, it suffices to show that there exists a point $(z, \bar{p}) \in A_r \times \bar{A}_r$ satisfying $K_{A_r}(z, \bar{p}) \neq 0$ and

$$K_{A_r}(z, \bar{p})^2 F_{A_r}(z, \bar{p}) = K_{A_r}(z, \bar{p})^4 \partial_z \bar{\partial}_w \Big|_{w=p} \log K_{A_r}(z, \bar{w}) = 0.$$

It is known that if r is sufficiently close to 0 and p is on the real axis, such a point (z, \bar{p}) exists, where z is near the imaginary axis (see [12], the proof of Theorem 1.5).

Moreover, this example can be modified to the case of irreducible strictly pseudoconvex domains as follows: Consider a strictly pseudoconvex exhaustion Ω_j for $A_r \times D$. Note that they are irreducible domains (cf. [19]). On the other hand, the Bergman kernel of Ω_j and its derivatives uniformly converge on compacta to those of $A_r \times D$ (cf. [26]). By Hurwitz's theorem, Ω_j satisfies $Z_0^p \subsetneq \widehat{Z}_1^p$ when j is large enough.

6. A GENERALIZATION OF THE LU THEOREM

We present an application of the Bochner connection. Let Ω be a bounded domain in \mathbb{C}^n and M a complex manifold with the positive-definite Bergman metric. Denote their Bergman metric by β_Ω and β_M , respectively. Call the point $p \in \Omega$ a *pole of the Bochner connection* ∇^p whenever $\text{rep}_p : \Omega^p \rightarrow \mathbb{C}^n$ is one-to-one.

Theorem 6.1. *Suppose that Ω has a pole p of ∇^p . If there is a surjective holomorphic map $f : \Omega \rightarrow M$ satisfying $f^* \beta_M = \beta_\Omega$, then f is a biholomorphism.*

This theorem is a generalization of the following well-known result.

Theorem 6.2 (Lu Qi-Keng [23]). *If Ω is a bounded domain in \mathbb{C}^n , whose Bergman metric is complete and has constant holomorphic sectional curvature, then Ω is biholomorphic to the unit ball.*

Proof of \lceil Theorem 6.1 \Rightarrow Theorem 6.2 \rfloor . Although the below proof is included in the proof of Theorem 4.2.2 in [14], we recall that for convinience. Let c be the constant, the holomorphic sectional curvature of Ω . If $c > 0$, then Ω would be a complete Riemannian manifold with all sectional curvatures $\geq c/4 > 0$. Thus Myers' theorem in Riemannian geometry implies that Ω is compact, a contradiction. If $c = 0$, then the covering space is \mathbb{C}^n . Therefore the covering map is constant by Liouville's theorem, which is impossible. Consequently, $c < 0$. In that case, it is known that the universal covering space of Ω is biholomorphic to the unit ball \mathbb{B}^n and the covering map $f : \mathbb{B}^n \rightarrow \Omega$ is a Bergman isometry. Therefore f has to be one-to-one by Theorem 6.1, and hence the conclusion of Theorem 6.2 follows. \square

Remark 6.3. Notice that Theorem 6.1 does not assume the completeness of the Bergman metric β_Ω . Moreover, the bounded domain Ω need not possess the constant holomorphic sectional curvature. Besides the unit ball, the following domains satisfy the hypothesis of Theorem 6.1.

- Every complete circular domain; the center is a pole.
- Every bounded homogeneous domain; every point is a pole (cf. [31]).

More generally, every bounded domain, which possesses a point p such that the matrix $G(z, \bar{p})$ is independent of z , satisfies the hypothesis (the point p is a pole). This is called a *representative domain* according to [24].

With slight modification of the proof of Theorem 4.2.2 in [14], we present

Proof of Theorem 6.1. We are only to show that f is one-to-one. Since $f^*\beta_M = \beta_\Omega$ implies that df is non-singular, f is locally invertible. Let V be a neighborhood of p and U a neighborhood of $q := f(p)$ such that $f|_V : V \rightarrow U$ is a biholomorphism. Denote by g_0 the inverse of $f|_V$.

On the other hand $\nabla_\Omega = f^*\nabla_M$, since $f^*\beta_M = \beta_\Omega$, where ∇ denotes the Bergman metric connection. The uniqueness of the polarization and the holomorphicity of f yield that $\nabla_\Omega^p = f^*\nabla_M^q$. This means that f maps geodesics of ∇^p to geodesics of ∇^q , and one sees that $A := \text{rep}_q \circ f|_V \circ \text{rep}_p^{-1}$ is \mathbb{C} -linear as in Remark 4.3. Thus $g_0 := f|_V^{-1} = \text{rep}_p^{-1} \circ A \circ \text{rep}_q$, where A is an invertible \mathbb{C} -linear map.

Note that $\text{rep}_q \circ f = A^{-1} \circ \text{rep}_p$ on $\Omega - (Z^p \cup f^{-1}(Z^q))$, where $Z^p := Z_0^p \cup Z_1^p$ and $Z^q := Z_0^q \cup Z_1^q$. Then the restriction map $\text{rep}_p^{-1}|_{A \circ \text{rep}_q(M - (f(Z^p) \cup Z^q))}$ is a well-defined holomorphic map. Since the linear map A is everywhere defined and rep_q extends to a holomorphic mapping of $M - Z^q$, so does g_0 . Denote by g the extension of g_0 .

Let $X := f^{-1}(Z^q)$. Then $g \circ f : \Omega - X \rightarrow \mathbb{C}^n$ is holomorphic and $g \circ f(z) = z$ for every $z \in \Omega - X$. Therefore, for every $\zeta \in M - Z^q$, choose $x \in \Omega$ such that $f(x) = \zeta$. Since $g(\zeta) = g(f(x)) = x$, $g(M - Z^q) \subset \Omega$. Note that g is a bounded holomorphic map on the connected manifold $M - Z^q$. By the Riemann extension theorem, g extends to a holomorphic mapping of M into \mathbb{C}^n . This shows that g is the left inverse to f , and hence f is one-to-one. \square

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